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Examples of Turbulence Models for Incompressible Flows

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The paper describes some currently available models for calculating turbulent stresses and heat or mass fluxes in incompressible flow which are more generally applicable than the Prandtl mixing-length hypothesis. These include models employing transport equations for the intensity and the length scale of the turbulent motion, notably the k-ϵ model, as well as second-order closure schemes based on transport equations for the turbulent stresses and heat or mass fluxes themselves. The individual models are introduced briefly, their merits and demerits are discussed, and typical examples of calculations relevant to aerospace problems are presented.

Nomenclature

c = constants in turbulence model
c_f = friction coefficient $= \tau_w / \frac{1}{2}\rho U_\infty^2$
c_p = specific heat
D, d = diameter or other geometrical parameters
E = friction parameter in law of the wall [Eq. (8)]
G = buoyancy production/destruction of $k = -\beta g_i \overline{u_i \phi}$
G_{ij} = buoyancy production/destruction of $\overline{u_i u_j} = -\beta (g_i \overline{u_j \phi} + g_j \overline{u_i \phi})$
g = gravitational acceleration
K = acceleration parameter $= \nu \mathrm{d}U_\infty / \mathrm{d}x / U_\infty^2$
k = turbulent kinetic energy $= \frac{1}{2}\overline{u_i u_i}$
L = length scale of turbulence
ℓ_m = Prandtl mixing length
P = stress production of $k = -\overline{u_i u_j} \partial U_i / \partial x_j$
P_{ij} = stress production of $\overline{u_i u_j} = -\overline{u_i u_\ell} \partial U_j / \partial x_\ell - \overline{u_j u_\ell} \partial U_i / \partial x_\ell$
q = heat flux at surface
R = radius
Re = Reynolds number
r = radial coordinate
St = Stanton number $= q / \rho c_p U_\infty (T_w - T_\infty)$
T = temperature
U, V, W = mean velocity components in x, y, z direction
u, v, w = fluctuating velocity components in x, y, z direction
U_i = mean-velocity component in x_i direction
u_i = fluctuating velocity component in x_i direction
U_τ = friction velocity $= \sqrt{\tau_w / \rho}$
U_∞ = freestream velocity
x, y, z = coordinates
x_i = coordinates in tensor notation
β = volumetric expansion coefficient
γ = constant in pressure-strain model
δ = shear-layer thickness
δ_{ij} = Kronecker delta = 1 for $i = j$ and = 0 for $1 \neq j$
ϵ = dissipation rate of k
κ = von Kármán constant
ν = kinematic molecular viscosity
ν_t = eddy (or turbulent) viscosity
ρ = fluid density
σ_t = turbulent Prandtl/Schmidt number
$\sigma_{k,\epsilon}$ = constants in k-ϵ model
τ_w = wall shear stress
Φ = mean scalar quantity
ϕ = fluctuating scalar quantity

Subscripts

j = jet
w = wall
0 = mainstream
∞ = condition in external (free) stream

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Introduction

IN spite of recent advances in the direct solution of the time-dependent Navier-Stokes equations and in large-eddy simulation techniques,[1,2] the only economically feasible way to solve practical turbulent flow problems is still the use of statistically averaged equations governing mean-flow quantities. In these equations, the transport of momentum, heat, and mass by turbulent motion is represented by correlations between fluctuating quantities. The momentum equations contain the turbulent or Reynolds stresses $-\rho\overline{u_i u_j}$, and the averaged scalar transport equation contains the turbulent heat or mass flux $-\rho\overline{u_i \phi}$, where u_i and u_j are fluctuating velocity components and ϕ is the fluctuating scalar quantity. Because of the appearance of these terms, the mean-flow equations are not closed, and a turbulence model is necessary to determine these turbulent transport terms before the equations can be solved.

One of the first turbulence models proposed, the Prandtl[3] mixing length hypothesis, is still among the most widely used models. It employs the eddy viscosity/diffusivity concept, which relates the turbulent transport terms to the gradients of mean-flow quantities. For thin shear layers, this concept reads:

$$-\overline{uv} = \nu_t \frac{\partial U}{\partial y} \qquad -\overline{v\phi} = \frac{\nu_t}{\sigma_t} \frac{\partial \Phi}{\partial y} \tag{1}$$

which by a modified Reynolds analogy assumes that the turbulent scalar transport is closely related to the momentum transport and σ_t is the turbulent Prandtl/Schmidt number. The Prandtl mixing-length hypothesis calculates the distribution of the eddy viscosity ν_t by relating it to the local mean-velocity gradient:

$$\nu_t = \ell_m^2 \left| \frac{\partial U}{\partial y} \right| \tag{2}$$

This relation, which is again written in a form suitable for thin shear layers, involves a single unknown parameter—the mixing length ℓ_m—whose distribution over the flowfield has to be prescribed with the aid of empirical information (see, e.g., Ref. 4). Only about 15 years ago, computers and numerical techniques became sufficiently advanced to solve the partial differential equations for flows of practical interest, and during this period the mixing-length model has been tested for and applied to a large variety of flow situations, most of them in the class of thin shear layers. In the process of this testing, great experience has been gathered in prescribing the mixing-length distributions for such layers, in particular for wall boundary layers (Patankar and Spalding,[5] Cebeci and Smith,[6] Crawford and Kays[7]), and much use has been made of the mixing-length model for practical calculations in the aerospace industry. This extensive testing has also brought to light the limitations of the mixing-length hypothesis, in particular the lack of universality of the empirical input. One of the main shortcomings of this hypothesis is that it is based on the implied assumption that turbulence is in local equilibrium, which means that, at each point in the flow, turbulence energy is dissipated at the same rate as it is produced, so that there can be no influence of turbulence production at other points in the flow or at earlier times. Hence, the mixing-length hypothesis cannot account for transport and history effects of turbulence. As a result, the mixing-length hypothesis predicts the eddy viscosity and diffusivity to be zero whenever the velocity gradient is zero, which leads to unrealistic simulations in many cases (see, e.g., Ref. 4).

A further disadvantage of the mixing-length model is that effects on the turbulence due to buoyancy, rotation, or streamline curvature can be accounted for in an entirely empirical way only, and it is difficult to devise generally applicable empirical relations for these effects. Although great experience has been gained on the distribution of the mixing length ℓ_m in simple shear layers, problems arise in determining ℓ_m in cases when several shear layers interact. The empirical specification for ℓ_m is particularly difficult, and sometimes even impossible, in flows that are more complex than shear layers, as for example in separated flows or in general three-dimensional flows.

In striving for more generally applicable models, turbulence models have been proposed that account for transport and history effects of turbulence by introducing transport equations for turbulence quantities. The present paper describes and compares several of these models. In order of increasing complexity, one-equation models employing a transport equation for the turbulent kinetic energy, the widely used k-ϵ model, and full as well as truncated second-order closure models based on transport equations for the individual Reynolds stresses and turbulent heat or mass fluxes are discussed briefly, with an emphasis on the latter two types of models because they are the only ones that are applicable to flows more complex than thin shear layers. For these models, examples of applications are presented whose successful calculation with the mixing-length hypothesis, if possible at all, would require empirical modification.

One-Equation Models

The simplest models accounting for the transport and history effects of turbulence use a transport equation for a suitable velocity scale of the turbulent motion. Usually as such a scale, $\sqrt{k}$ is taken, where k is the kinetic energy of the turbulent motion, which is a measure of the intensity of the turbulent fluctuations in the three directions. In most models, the following transport equation for k is employed:

$$\underbrace{\frac{\partial k}{\partial t}}_{\text{rate of change}} + \underbrace{U_i \frac{\partial k}{\partial x_i}}_{\text{convection}} = \underbrace{\frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}\right)}_{\text{diffusion}} \underbrace{- \overline{u_i u_j}\frac{\partial U_i}{\partial x_j}}_{P=\text{production}} \underbrace{- \epsilon}_{\text{dissipation}} \qquad (3)$$

This equation is exact as derived from the Navier-Stokes equations except for the diffusion term, in which the diffusion flux of k has been assumed to be proportional to the gradient of k (σ_k is an empirical constant). Equation (3) describes how the rate of change of k is balanced by convective transport by the mean motion, diffusive transport by turbulent motion, production by interaction of turbulent stresses and mean-velocity gradients, and destruction by the dissipation ϵ. In buoyant flows, an additional buoyancy production/destruction term is present.[4]

In one-equation models, the dissipation rate ϵ is usually determined from

$$\epsilon = c_D (k^{3/2}/L) \qquad (4)$$

which is an outcome of dimensional analysis when one assumes that the dissipation rate ϵ is governed by large-scale turbulent motion and that this motion is characterized by the velocity scale $\sqrt{k}$ and the length scale L. Two principal suggestions have been made to relate the turbulent stresses to the kinetic energy k determined from a transport equation. The first suggestion employs the eddy-viscosity concept, in which case dimensional analysis yields the so-called Kolmogorov[8]-Prandtl[9] expression

$$\nu_t = c'_\mu \sqrt{k} L \qquad (5)$$

The other suggestion is due to Bradshaw[10] and his co-workers, who did not employ the eddy-viscosity concept but converted the kinetic-energy equation into a transport equation for the shear stress $\overline{uv}$ by assuming a direct link between $\overline{uv}$ and k. The original model was intended only for wall boundary layers where experiments suggest that $\overline{uv} \sim 0.3\,k$. Bradshaw et al.[10] further did not approximate the diffusion of k by a gradient-diffusion model but assumed instead that the diffusion flux of k is proportional to a bulk velocity.

In one-equation models, the length scale L appearing in Eqs. (4) and (5) is usually determined from simple empirical relations similar to those used for the mixing length ℓ_m. This empirical specification works quite well for simple shear layers, and therefore Bradshaw et al.'s[10] method was used with great success in many wall boundary-layer calculations. In complex flows, however, L is no easier to prescribe than the mixing length ℓ_m. For this reason, the application of one-equation models was limited mainly to shear-layer flows. Various authors have tried to develop formulas for calculating L in general flows. A discussion on this can be found in Rodi,[4] who concluded that these formulas were insufficiently tested and also rather complex and expensive of computing time. Hence, the trend has been to use two-equation models that also determine the length scale from a transport equation.

One-equation models using the k equation (3) and c'_μ and c_D in Eqs. (4) and (5) as fixed empirical constants are applicable only to flows or flow regions where the local turbulent Reynolds number $Re_T = \nu_t/\nu$ is sufficiently high. Hence, they are not applicable very near walls, and the viscous sublayer has to be bridged by wall functions, as given in the next section. Hassid and Poreh[11] and Norris and Reynolds[12,13] have proposed low-Reynolds-number versions of one-equation models that allow an integration right to the wall. In these models, an additional, exact, viscous diffusion term is included in the k equation and some of the empirical constants are replaced by functions of the turbulent Reynolds number Re_T. Such models work well in boundary layers with zero and adverse pressure gradients. Hassid and Poreh's model, however, gives fairly poor predictions for strongly accelerating boundary layers (see Fig. 5 below), and Norris and Reynolds report that such flows could be predicted well only when an empirical modification to the length scale is introduced which accounts for the influence of the acceleration.

The k-ϵ Model

The length scale of the turbulent motion is subject to transport and history processes in a similar way to the turbulent energy k. In order to account for these processes, models were suggested that use a transport equation for the length scale L, and the difficulties in finding generally valid formulas for prescribing or calculating L have stimulated the use of such equations. The dependent variable of the length-scale-determining equation must not, however, be the length scale L itself; any combination with k will suffice since k is known from the solution of the k equation. Equations for various combinations have been proposed, but the k-ϵ model using an equation for $\epsilon \propto k^{3/2}/L$ has become most popular, mainly because of the practical advantage over other equations that the ϵ equation requires no extra terms near walls. Because of its popularity, discussion is restricted here to the k-ϵ model.

The k-ϵ model employs the eddy viscosity/diffusivity concept (given below for general flows) and relates the eddy viscosity ν_t to k and ϵ via the Kolmogorov-Prandtl relation [Eq. (5)] (noting that $\epsilon \propto k^{3/2}/L$):

$$-\overline{u_i u_j} = \nu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij} \qquad \nu_t = c_\mu \frac{k^2}{\epsilon} \tag{6}$$

The distribution of k is determined from Eq. (3) and that of ϵ from the following transport equation:

$$\frac{\partial \epsilon}{\partial t} + U_i \frac{\partial \epsilon}{\partial x_i} = \frac{\partial}{\partial x_i} \left(\frac{\nu_t}{\sigma_\epsilon} \frac{\partial \epsilon}{\partial x_i} \right) + c_{\epsilon 1} \frac{\epsilon}{k} P - c_{\epsilon 2} \frac{\epsilon^2}{k} \tag{7}$$

An exact ϵ equation can be derived from the Navier-Stokes equations, but such drastic model assumptions have to be introduced that the resulting ϵ equation has highly empirical character. The terms in the ϵ equation (7) represent physical processes similar to those discussed already for the k equation; that is, the rate of change of ϵ is balanced by convective and diffusive transport and by production and destruction processes.

The k-ϵ model contains the five empirical constants c_μ, σ_k, σ_ϵ, $c_{\epsilon 1}$, $c_{\epsilon 2}$, and, when the turbulent heat and mass transfer is to be calculated, also the turbulent Prandtl/Schmidt number σ_t. The standard values for these constants and also a more detailed discussion of the k-ϵ model are given in Refs. 4 and 14. The standard model presented so far is applicable only to flows or flow regions with high turbulent Reynolds number $Re_T = \nu_t/\nu$ and cannot be applied near walls, where viscous effects become dominant. Hence, the so-called wall-function approach is used in order to bridge the viscous sublayer. This approach assumes that, at a point with wall distance y_c just outside the viscous sublayer, the velocity components parallel to the wall follow the logarithmic law of the wall and the turbulence is in local equilibrium, so that the production P is equal to the dissipation ϵ. With these assumptions, the resultant velocity parallel to the wall U_c, the kinetic energy k_c, and the dissipation rate ϵ_c at point y_c are usually related to the resultant friction velocity U_τ by the following relations

$$\frac{U_c}{U_\tau} = \frac{1}{\kappa} \ell n \left(E \frac{U_\tau y_c}{\nu} \right) \qquad k_c = \frac{U_\tau^2}{\sqrt{c_\mu}} \qquad \epsilon_c = \frac{U_\tau^3}{\kappa y_c} \tag{8}$$

Over the last 10 years, the k-ϵ model has been applied to a large number of different flows so that it is now one of the best-tested turbulence models. In the course of testing it has been found to be significantly more general than the mixing-length hypothesis. With the same empirical input, the model can simulate many free shear layers[15] as well as wall boundary layers and duct flows. An example for a free-shear-layer calculation is given in Fig. 1, which shows the development of the velocity profile in a mixing layer from wall boundary-layer profiles at the end of a splitter plate. Calculations are included as obtained with the mixing-length hypothesis, a one-equation model using the Kolmogorov-Prandtl Eq. (5) with L proportional to the local shear-layer width, and the k-ϵ model, all starting from the same initial profile. As can be seen, development of the velocity profile is described best by the k-ϵ model. The less satisfactory results obtained with the other two models point to the difficulties in specifying the length-scale distribution in situations where two shear layers interact as do the two wall boundary layers in this case.

Scheuerer[17] used the k-ϵ model to calculate wall boundary layers under the influence of freestream turbulence, and Fig. 2

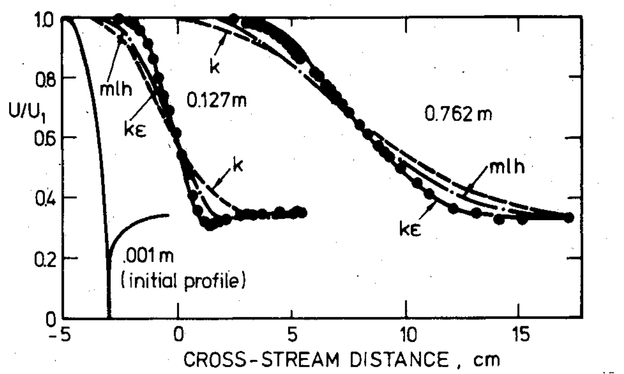

Fig. 1 Development of a plane mixing layer, —— = predictions;[15] ● = measurements.[16]

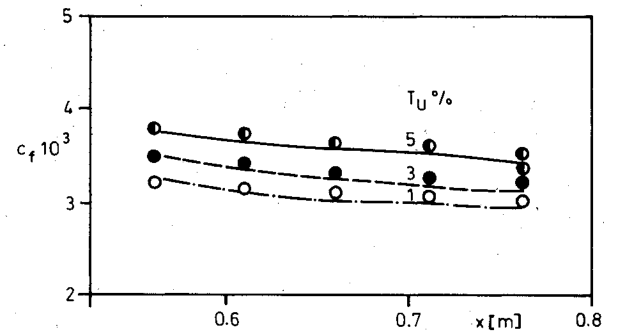

Fig. 2 Development of friction coefficient in boundary layers with freestream turbulence; ——, ——, —·— = predictions;[17] ◐, ●, ○ = measurements.[18]

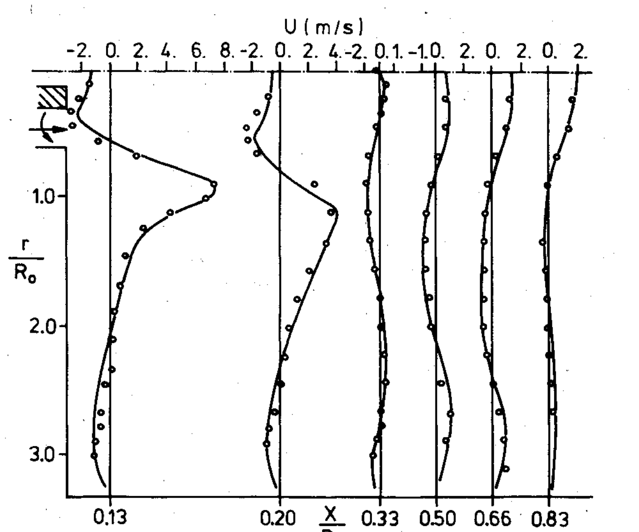

Fig. 3 Velocity profiles in model combustion chamber; —— = predictions; ○ = measurements (from Ref. 20).

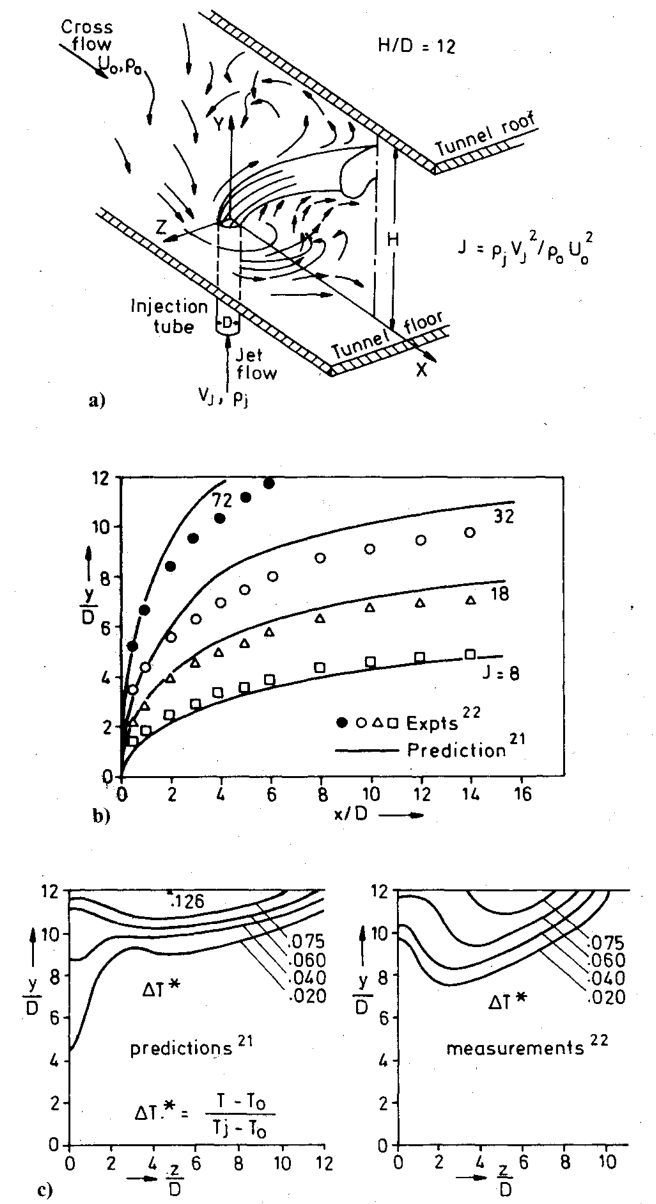

Fig. 4 Round jet in bounded cross flow (from Ref. 21); a) flow configuration; b) jet trajectory; c) temperature contours for $J = 72$ at $x/D = 12$.

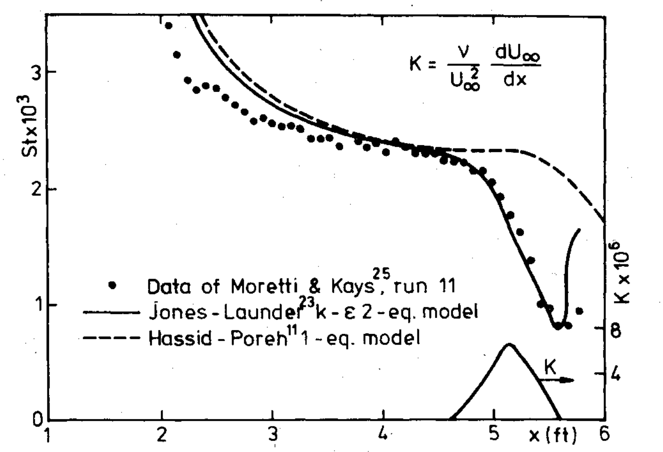

Fig. 5 Variation of Stanton number in an accelerated boundary layer.

shows that the friction coefficient c_f under such influence is well predicted. It is important to note that the experimentally observed freestream turbulence entered the calculations only in the form of boundary conditions for k and ϵ but that the turbulence model itself was unchanged. In contrast, the simulation of freestream turbulence effects with the mixing-length hypothesis would require empirical adjustments of the model. In practical calculations, errors in c_f of up to 10% must be expected when such adjustments are not made. One-equation models would, however, give results similar to those obtained with the k-ϵ model.

Many successful applications of the k-ϵ model to separating flows have been reported in the literature, but it should be mentioned that success was much better for confined flows and that the calculations were not always satisfactory for unconfined, separated flows (e.g., near wake behind a disk). It should also be mentioned that in separated flow calculations the results are often subject to numerical inaccuracies,[19] so that it is not the turbulence model alone that is responsible for lack of agreement with experimental data. An example of a confined flow calculation is given in Fig. 3, which shows predicted and measured velocity profiles in a model combustion chamber. Here the simulation is of an experimental situation without combustion, where all the fluid entered through the annular inlet with a swirl component superposed. The development of this rather complex axisymmetric flow can be seen to be well simulated by the k-ϵ model. Even more complex three-dimensional flows have also been calculated with the k-ϵ model, and Fig. 4 shows the application to a jet in a bounded cross flow. Predictions are given for various ratios J of jet-to-cross-flow momentum, and for one such ratio the temperature distribution is shown for a cross section downstream of the jet discharge. The agreement between predictions and measurements is not perfect, particularly for the higher values of the parameter J, but in view of the fact that the calculations were influenced somewhat by numerical inaccuracies,[21] the results for such a complex flow are still encouraging because the qualitative agreement with the experiments is good and the gross features of the flow are well predicted.

Compilations of further examples of the application of the k-ϵ model can be found in Refs. 4, 14, and 15. These papers also discuss a few exceptional flow situations that cannot be calculated satisfactorily with the standard constants in the model, notably so-called weak free shear layers (e.g., far wakes), where overall turbulence production is small compared with dissipation, and axisymmetric jets. The references give empirical functions to replace some of the constants so that these special flows can also be predicted with the k-ϵ model.

So far the discussion has been confined to the high-Reynolds-number version of the k-ϵ model, which cannot be applied in the immediate vicinity of a wall. Jones and Launder[23] have proposed a low-Reynolds-number version with which the calculations can be carried right to the wall. In this model, viscous diffusion of k and ϵ is included in Eqs. (3) and (7). Effectively this means that ν_t/σ_k and ν_t/σ_ϵ are enhanced by the molecular viscosity ν in the first terms on the right-hand side of Eqs. (3) and (7). Further, the constants c_μ and $c_{\epsilon 2}$ are replaced by functions of the turbulent Reynolds number $Re_T = \nu_t/\nu$, and an additional term is added to the ϵ equation in order to obtain a realistic k distribution in the viscous sublayer. For flow regions with high Re_T, the model is identical to the high-Reynolds-number k-ϵ model discussed above. Jones and Launder applied their model successfully to calculate the near-wall region in high-Reynolds-number pipe flow, the friction coefficient and the velocity profile in low-Reynolds-number (but still fully turbulent) pipe flow, and various strongly accelerating boundary layers such as sink flow in which relaminarization occurred. Gibson[24] has made heat-transfer calculations for the accelerating boundary layer investigated by Moretti and Kays[25] with both the Jones-Launder k-ϵ model and the one-equation model of Hassid and Poreh.[11] The predicted variation of the Stanton number along the plate is compared with the measurements in Fig. 5, where also the variation of the acceleration parameter K is given. The fall in Stanton number in the accelerated flow and the subsequent recovery are simulated well by the k-ϵ model, whereas Hassid and Poreh's one-equation model fails to reproduce these effects. The predicted Stanton number upstream of the acceleration is somewhat too high because, for simplicity, the change in wall temperature at $x=2.1$ ft has been taken as a step change rather than as the gradual decrease that occurred in the experiment. An important conclusion from Fig. 5 is that acceleration considerably influences the length scale in a turbulent boundary layer and that this influence is simulated rather well by the k-ϵ two-equation model without additional empirical input, whereas one-equation models would require such input (see also Ref. 12). A comparison of the performance of eight low-Reynolds-number two-equation models can be found in Ref. 26.

Turbulent-Stress/Flux Equation Models

The k-ϵ model is based on the eddy viscosity/diffusivity concept, which is not valid under all circumstances. A greater practical limitation is, however, that eddy viscosity and diffusivity are assumed to be isotropic; that is, the same values are taken for the various $\overline{u_iu_j}$'s and $\overline{u_i\phi}$'s. In complex flows, eddy viscosity and diffusivity will certainly depend on the stress or flux component considered, and the same is true when turbulence is strongly influenced by body forces acting in a preferred direction, such as buoyancy forces. Further, all models discussed so far assumed that the local state of turbulence can be characterized by one velocity scale $\sqrt{k}$ and that the individual $\overline{u_iu_j}$'s can be related to this scale. In reality, however, the individual $\overline{u_iu_j}$'s may develop quite differently in the flow, and when this is the case, the models discussed so far will be too simple. In order to account for the different development of the individual stresses, transport equations for $\overline{u_iu_j}$ have been introduced and analogous equations also for the scalar fluxes $\overline{u_i\phi}$. These equations can be derived in exact forms, but they contain higher-order correlations that have to be approximated in order to obtain a closed system. A particular advantage of deriving the exact equations is that terms accounting for buoyancy, rotation, and other effects are introduced automatically.

Models employing transport equations for $\overline{u_iu_j}$ and $\overline{u_i\phi}$ are often called second-order-closure schemes, and a variety of such models have been proposed in the literature. Here only one relatively simple model will be presented: that developed by Launder and his co-workers,[27,28] which has been tested already by application to a variety of flows. Somewhat different models have been developed by Lumley and his co-workers[29] and by Donaldson and his associates.[30] Launder et al.[27] and Gibson and Launder[28] proposed, respectively, the following modeled transport equations for $\overline{u_iu_j}$ and $\overline{u_i\phi}$:

$$\underbrace{\frac{\partial \overline{u_iu_j}}{\partial t}}_{\text{rate of change}} + \underbrace{U_\ell \frac{\partial \overline{u_iu_j}}{\partial x_\ell}}_{\text{convective transport}} = \underbrace{c_s \frac{\partial}{\partial x_\ell}\left(\frac{k}{\epsilon}\overline{u_ku_\ell}\frac{\partial \overline{u_iu_j}}{\partial x_k}\right)}_{\text{diffusive transport}} \underbrace{- \overline{u_iu_\ell}\frac{\partial U_j}{\partial x_\ell} - \overline{u_ju_\ell}\frac{\partial U_i}{\partial x_\ell}}_{P_{ij}=\text{stress production}}$$

$$\underbrace{-c_1\frac{\epsilon}{k}\left(\overline{u_iu_j}-\frac{2}{3}\delta_{ij}k\right) - c_2\left(P_{ij}-\frac{2}{3}\delta_{ij}P\right) - c_3\left(G_{ij}-\frac{2}{3}\delta_{ij}G\right)}_{\text{pressure strain}}$$

$$\underbrace{-\beta(g_i\overline{u_j\phi}+g_j\overline{u_i\phi})}_{G_{ij}=\text{buoyancy production}} \underbrace{-\frac{2}{3}\epsilon\delta_{ij}}_{\text{viscous dissipation}} \qquad (9)$$

$$\underbrace{\frac{\partial \overline{u_i\phi}}{\partial t}}_{\text{rate of change}} + \underbrace{U_\ell \frac{\partial \overline{u_i\phi}}{\partial x_\ell}}_{\text{convective transport}} = \underbrace{c_{s\phi} \frac{\partial}{\partial x_\ell}\left(\frac{k}{\epsilon}\overline{u_ku_\ell}\frac{\partial \overline{u_i\phi}}{\partial x_k}\right)}_{\text{diffusive transport}} \underbrace{- \overline{u_iu_j}\frac{\partial \Phi}{\partial x_j} - \overline{u_j\phi}\frac{\partial U_i}{\partial x_j}}_{\text{mean-flow production}}$$

$$\underbrace{-\beta g_i\overline{\phi^2}}_{\text{buoyancy production}} \underbrace{-c_{1\phi}\frac{\epsilon}{k}\overline{u_i\phi} + c_{2\phi}\overline{u_\ell\phi}\frac{\partial U_i}{\partial x_\ell} + c_{3\phi}\beta g_i\overline{\phi^2}}_{\text{pressure scrambling}} \qquad (10)$$

The physical interpretation of the individual terms is indicated in the equations. The rate of change, convective transport, and mean-field as well as the buoyancy production terms are exact, whereas the diffusion, pressure-strain/scrambling, and viscous dissipation terms are model approximations. The diffusion fluxes of $\overline{u_iu_j}$ and $\overline{u_i\phi}$ have been expressed by simple gradient-diffusion models. Local isotropy has been assumed to prevail so that the dissipation is the same for all three normal-stress components (and thus ⅔ of the total dissipation ϵ) and so that the viscous destruction terms for shear stresses and also for scalar fluxes originally appearing in the $\overline{u_i\phi}$ equation are zero. The most important approximations concern the pressure-strain/scrambling terms since, for shear stresses and scalar fluxes, these are the main terms to balance the production of these quantities. The pressure-strain/scrambling model can be seen to consist of three parts, the first one representing the interaction of fluctuating quantities only, the second the interaction of mean-strain and fluctuating quantities, and the third the effect of buoyancy forces. The pressure-strain/scrambling model presented here is the simplest one that accounts for all three mechanisms. More complex models have been proposed but, except under extreme circumstances, have not been found superior. In buoyant situations the $\overline{u_i\phi}$ equation (10) involves the fluctuating scalar $\overline{\phi^2}$, which in a second-order-closure scheme is also determined from a transport equation as given in Refs. 4 and 28. The dissipation rate ϵ appearing in Eqs. (9) and (10) is determined from the ϵ equation (7), with the diffusion expression replaced by the more general relation

$$\text{diffusion}_\epsilon = c_\epsilon \frac{\partial}{\partial x_\ell}\left(\frac{k}{\epsilon}\overline{u_\ell u_k}\frac{\partial \epsilon}{\partial x_k}\right) \qquad (11)$$

which is in line with the diffusion expressions used in Eqs. (9) and (10).

In local-equilibrium shear layers, where the rate-of-change, convection, and diffusion terms in Eq. (9) are absent, the ratio of the individual stresses $\overline{u_iu_j}$ to one another (and thus to the kinetic energy k) are determined solely by the pressure-strain model. Experiments have shown that these ratios are significantly different in shear layers near to and remote from walls: in near-wall turbulence, the level of the fluctuating velocity normal to the surface is much damped whereas that parallel to the main flow is enhanced relative to shear flows without the influence of a wall. The pressure-strain model introduced thus far does not produce these differences because it does not account for any wall effects. Hence, a wall correction must be introduced to the pressure-strain model and, by analogy, also to the pressure-scrambling model in Eq. (10). Various proposals for such corrections have been made. For example, Launder, Reece, and Rodi[27] basically make the empirical constants in the pressure-strain model a function of the relative distance from the wall, $L/y \propto k^{3/2}/(\epsilon y)$. The details of this approach can be found in Ref. 27. The empirical constants appearing in the above $\overline{u_iu_j}$ and $\overline{u_i\phi}$ equations and in the wall corrections are compiled in Ref. 4. Some examples of calculations with stress-equation models will be presented below together with results obtained with algebraic stress models.

Algebraic Stress/Flux Models

Models employing transport equations for the individual turbulent stresses and fluxes constitute a quite large number of differential equations whose solution is not a trivial task and is also rather costly. Hence, for practical applications, it would be desirable to use simplified models whenever possible. For this reason, so-called algebraic stress/flux models have been developed by simplifying the differential transport equations such that they reduce to algebraic expressions but retain most of their basic features. Since it is the convection and diffusion terms that make the transport equations differential equations, these terms need to be simplified by model approximations. The simplest approximation is of course to neglect these terms, but a more generally valid approximation was proposed by Rodi,[31] who assumed that the transport of $\overline{u_iu_j}$ is proportional to the transport of k (which is $P+G-\epsilon$) and that the proportionality factor is the ratio $\overline{u_iu_j}/k$. With such approximations introduced, the transport equations yield algebraic expressions for $\overline{u_iu_j}$ and $\overline{u_i\phi}$ that contain the various production terms appearing in the $\overline{u_iu_j}$ and $\overline{u_i\phi}$ equations, respectively, and thus the gradients of the mean-flow quantities. k and ϵ appear also in the expressions, so that the k and ϵ equations (3) and (7) have to be added in order to complete the turbulence model. The algebraic expressions together with the k and ϵ equations hence form an extended k-ϵ model. The actual expressions are given in Ref. 4, where also a detailed discussion on algebraic stress/flux modeling is presented.

Algebraic stress/flux models are suitable whenever the transport of $\overline{u_iu_j}$ and $\overline{u_i\phi}$ is not very important, as this has been either neglected or modeled rather crudely. The algebraic stress relations are basically eddy viscosity formulas and are therefore not applicable to countergradient-diffusion situations that occur, for example, in the atmospheric boundary layer. However, in flows associated with engineering equipment, such situations are usually limited to small areas so that they are not of great practical relevance, as will be shown below. On the other hand, all effects that enter the transport equations for $\overline{u_iu_j}$ and $\overline{u_i\phi}$ through source terms can be accounted for by an algebraic stress/flux model, as, for example, body force effects (buoyancy, rotation, streamline curvature), nonisotropic strain fields, and wall-damping influence. Algebraic stress/flux models can therefore also simulate many of the flow phenomena that were described successfully by stress-equation models. Some examples will now be given.

The first example concerns the calculation of developed flow in a square duct, where gradients of the Reynolds stresses in the cross-sectional plane cause a secondary motion that is not present under laminar conditions and is hence often labeled "turbulence-driven." This motion cannot be predicted with an isotropic eddy viscosity as employed in the standard k-ϵ model (in laminar flow the viscosity is isotropic). A realistic simulation of the individual Reynolds stresses is required in this case, and Reece[32] and Naot and Rodi[33] have shown that the Launder, Reece, and Rodi[27] model is capable of doing this, but only with a somewhat more complex pressure-strain model than that given in Eq. (9). Reece employed differential stress equations whereas Naot and Rodi used an algebraic stress model. Figure 6 compares the longitudinal velocity contours and secondary velocity distributions predicted by both models with measurements. The bulging of the velocity contours toward the corner, which is due to the secondary motion, is reasonably well predicted by both models and so are the secondary velocity profiles. Differential and algebraic stress models yield about the same degree of agreement with the experiments in this case.

As a second example, calculations are presented for the curved mixing layer studied experimentally by Castro and

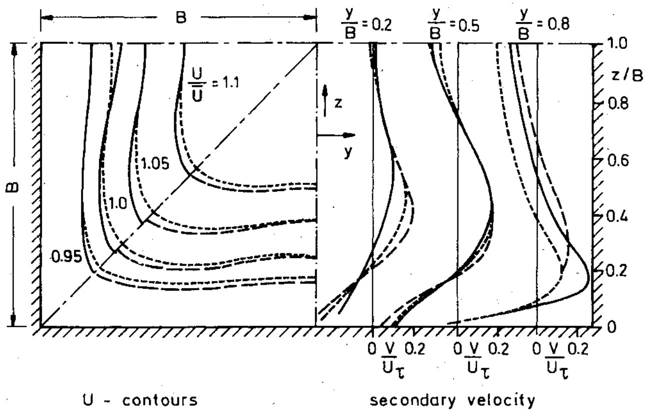

Fig. 6 Developed flow in square channel; — algebraic stress model;[33] —— stress-equation model;[32] ---experiments.[34]

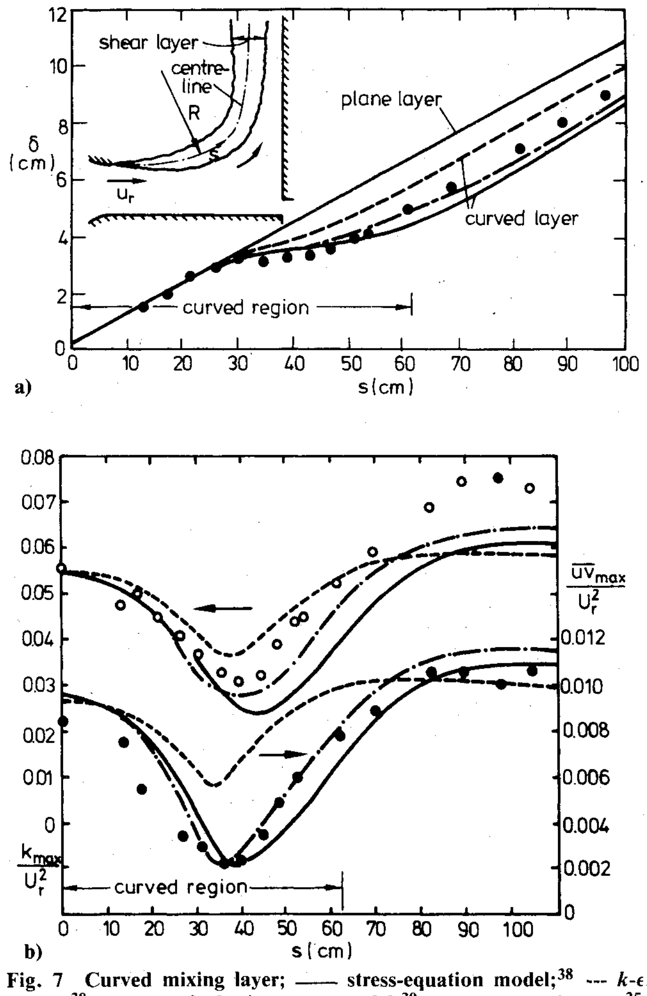

Fig. 7 Curved mixing layer; —— stress-equation model;[38] --- k-ϵ model;[38] — — — algebraic stress model;[39] ○, ● experiments.[35] a) growth of a layer width δ; b) variation of maximum k and $\overline{uv}$.

Bradshaw.[35] In the calculations, the curvature effects enter automatically in the $\overline{u_iu_j}$ equations by writing these in a curved coordinate system with the longitudinal coordinate s along the centerline of the curved mixing layer (see Fig. 7a) and the lateral coordinate normal to s. The calculations were carried out in order to test whether the stress-equation and algebraic stress models, without modification other than expressing the equations in a curved coordinate system, can predict the rather strong effects of streamline curvature on turbulence observed in the experiments. Figure 7 shows predictions obtained with three different turbulence models, one being the standard k-ϵ model (in a curved coordinate system, a curvature term appears in the k equation), the second the stress-equation model using Eq. (9), and the third an algebraic stress model that uses truncated forms of Eq. (9). Figure 7a demonstrates that the growth of the layer width δ (for a definition see Ref. 35) is reduced significantly by the curvature and that this is simulated well by the stress-equation and algebraic stress models. In contrast, the standard k-ϵ model does not respond sufficiently to the curvature. Figure 7b presents the streamwise variation of the maximum shear stress and kinetic energy. Both are reduced by the curvature, but $\overline{uv}_{max}$ is reduced more so that the structure parameter $\overline{uv}/k$ is also decreased by the curvature. The stress-equation and algebraic stress models simulate the influence of curvature on both $\overline{uv}$ and k reasonably well, whereas the standard k-ϵ model predicts only the streamwise variation of k fairly well but badly underpredicts the fall of $\overline{uv}$ and thus does not describe the observed behavior of the structure parameter $\overline{uv}/k$. It can be concluded that it is important to account for the influence of curvature on the individual stresses and not only on k and that both the stress-equation and algebraic stress models do this successfully.

To conclude this section, two further examples of calculations with algebraic stress/flux models will be presented. The first example is a plane wake in stably stratified media. An experiment was simulated in which a ribbon was pulled through water having a linear stable stratification, that is, where $d\rho/dy$ was constant and negative. The vertical turbulent fluctuations in the wake are damped by the stable stratification, and the growth of the wake is thereby strongly reduced. Hossain[38] has calculated this flow with an algebraic stress model that is basically a k-ϵ model with the empirical coefficients c_μ and σ_t replaced by functions of some local buoyancy parameter, where these functions are provided by the algebraic stress/flux relations. Figure 8 shows that the model simulates very well the reduction in wake spreading due to the stable stratification. For comparison, results are included also of a calculation of a wake in an unstratified medium, and here the model predicts the standard growth of the wake as $x^{1/2}$.

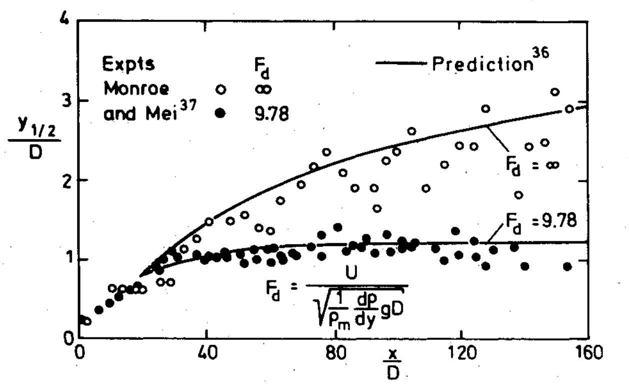

Fig. 8 Growth of half-width $y_{1/2}$ of plane wake in neutrally and stably stratified medium.

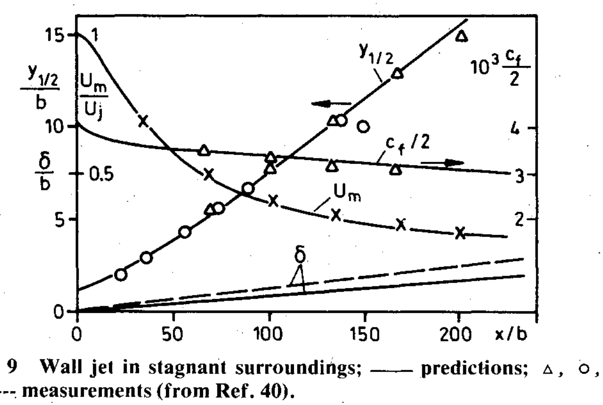

Fig. 9 Wall jet in stagnant surroundings; —— predictions; △, ○, ×, --- measurements (from Ref. 40).

The last example is a plane wall jet in stagnant surroundings. For this flow the standard k-ϵ model predicts the spreading rate (growth of the half width $y_{1/2}$) too high by about 30%. The reason is that this model does not account for the wall damping of the lateral fluctuations. As this damping influence is felt not only in the near-wall layer but also in the free shear layer beyond the velocity maximum, it is responsible for the reduced spreading rate compared with that of a free jet. When the wall damping is accounted for by a wall correction to the pressure-strain model and when this is retained in the process of reducing the $\overline{u_i u_j}$ equations to algebraic expressions, the correct wall-jet spreading is predicted, as can be seen from Fig. 9, where the growth of the half-width $y_{1/2}$ is compared with measurements. The streamwise variation of maximum velocity and wall shear stress is also well predicted, but the location δ of the velocity maximum is too close to the wall because in this flow the locations of zero shear stress and zero velocity gradient do not coincide, whereas the algebraic stress model (being of the eddy viscosity type) predicts zero shear stress at the location of the velocity maximum. The difference in locations can be predicted only by use of a full stress-equation model,[32] but it is of little practical importance. The same feature can be found in asymmetric channel flow with one rough and one smooth wall and was predicted well with the stress-equation model of Launder, Reece, and Rodi.[27] Again, the separation between the locations of zero shear stress and zero velocity gradient is of little practical significance because it is only about 0.05 of the channel width. Hence, an eddy viscosity model would yield results of an accuracy sufficient for engineering purposes also in this case.

In wall jets on convex surfaces, the locations of zero shear stress and zero velocity gradient move apart with increasing curvature, and the stress-equation model[41] predicts this flow correctly whereas preliminary calculations with the algebraic stress model indicate the limits of this model for cases with strong curvature. The stress-equation model, which accounts for curvature in the same way as was discussed above for the curved mixing layer, simulates correctly the increase of jet spreading due to the destabilizing effect of the curvature.

Conclusions

In the past, the mixing-length model has been used widely and with considerable success for calculations of simple shear layers, and a great amount of experience has been collected on the empirical specification of the mixing-length distribution in such flows. The mixing-length hypothesis is, however, not suitable whenever turbulence transport and history effects are important, and it is of little use for flows more complex than shear layers because of the great difficulties in specifying the mixing-length distribution in such flows. Further, extra effects on turbulence, such as those due to body forces, can be accounted for in an entirely empirical way only. One-equation models employing a transport equation for the kinetic energy of turbulence account for transport and history effects and are therefore superior to the mixing-length hypothesis for such nonequilibrium shear layers where the length-scale distribution can be prescribed realistically; they are, however, not very suitable for complex flows where an empirical length-scale determination is difficult. The simplest models for calculating such flows are two-equation models employing an additional transport equation for the length scale. Among these, the k-ϵ model has been tested most widely and has been shown to predict, with the same empirical input, many different flows, including separating and complex three-dimensional flows, with an accuracy sufficient for practical purposes. The eddy viscosity concept itself and, more important, the use of an isotropic eddy viscosity in the k-ϵ model do not, however, describe certain important flow phenomena.

Stress/flux equation models simulate the turbulence processes more realistically and are capable of describing many of the features that defy simulation with an isotropic eddy viscosity model. Further, extra effects such as those due to body forces enter automatically as a consequence of the derivation of the stress and flux equations. Models of this type are rather complex and computationally expensive so that they are not very suitable for practical applications. They are important, however, as starting points for deriving algebraic stress/flux expressions that are able to account for most of the effects described by the full equations. There seem to be only a few engineering problems where a full transport-equation model is needed.

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